

EXPONENTIAL CARMICHAEL FUNCTION

ANDREW V. LELECHENKO

ABSTRACT. Consider exponential Carmichael function $\lambda^{(e)}$ such that $\lambda^{(e)}$ is multiplicative and $\lambda^{(e)}(p^a) = \lambda(a)$, where λ is usual Carmichael function. We discuss the value of $\sum \lambda^{(e)}(n)$, where n runs over certain subsets of $[1, x]$, and provide bounds on the error term, using analytic methods and especially estimates of $\int_1^T |\zeta(\sigma + it)|^m dt$.

1. INTRODUCTION

Consider an operator E over arithmetic functions such that for every f the function Ef is multiplicative and

$$(Ef)(p^a) = f(a), \quad p \text{ is prime.}$$

For various functions f (such as the divisor function, the sum-of-divisor function, Möbius function, the totient function and so on) the behaviour of Ef was studied by many authors, starting from Subbarao [13]. The bibliography can be found in [11].

The notation for Ef , established by previous authors, is $f^{(e)}$.

Carmichael function λ is an arithmetic function such that

$$\lambda(p^a) = \begin{cases} \phi(p^a), & p > 2 \text{ or } a = 1, 2, \\ \phi(p^a)/2, & p = 2 \text{ and } a > 2, \end{cases}$$

and if $n = p_1^{a_1} \cdots p_m^{a_m}$ is a canonical representation, then

$$\lambda(n) = \text{lcm}(\lambda(p_1^{a_1}), \dots, \lambda(p_m^{a_m})).$$

This function was introduced at the beginning of the XX century in [2], but intense studies started only in 1990-th, e. g. [3]. Carmichael function finds applications in cryptography, e. g. [4].

Consider also the family of multiplicative functions

$$\delta_r(p^a) = \begin{cases} 0, & a < r, \\ 1, & a \geq r, \end{cases} \quad r \text{ is integer.}$$

Function δ_2 is a characteristic function of the set of square-full numbers, δ_3 — of cube-full numbers and so on. Of course, $\delta_1 \equiv 1$.

Denote $\lambda_r^{(e)}$ for the product of δ_r and $\lambda^{(e)}$:

$$\lambda_r^{(e)}(n) = \delta_r(n) \lambda^{(e)}(n).$$

The aim of our paper is to study asymptotic properties of $\lambda^{(e)} \equiv \lambda_1^{(e)}, \lambda_2^{(e)}, \lambda_3^{(e)}$ and $\lambda_4^{(e)}$.

Note that all proofs below remains valid for $\phi_r^{(e)}(n) = \delta_r(n) \phi^{(e)}(n)$ instead of $\lambda_r^{(e)}(n)$ for $r = 1, 2, 3, 4$.

2010 *Mathematics Subject Classification.* 11A25 11M06, 11N37, 11N56.

Key words and phrases. Exponential divisors, Carmichael function, moments of Riemann zeta-function.

2. NOTATIONS

Letter p with or without indexes denotes a prime number.

We write $f \star g$ for Dirichlet convolution

$$(f \star g)(n) = \sum_{d|n} f(d)g(n/d).$$

Denote

$$\tau(a_1, \dots, a_k; n) := \sum_{d_1^{a_1} \dots d_k^{a_k} = n} 1.$$

In asymptotic relations we use \sim , \asymp , Landau symbols O and o , Vinogradov symbols \ll and \gg in their usual meanings. All asymptotic relations are given as an argument (usually x) tends to the infinity.

Everywhere $\varepsilon > 0$ is an arbitrarily small number (not always the same even in one equation).

As usual $\zeta(s)$ is Riemann zeta-function. Real and imaginary components of the complex s are denoted as $\sigma := \Re s$ and $t := \Im s$, so $s = \sigma + it$.

For a fixed $\sigma \in [1/2, 1]$ define

$$m(\sigma) := \sup \left\{ m \mid \int_1^T |\zeta(\sigma + it)|^m dt \ll T^{1+\varepsilon} \right\}.$$

and

$$\mu(\sigma) := \limsup_{t \rightarrow \infty} \frac{\log |\zeta(\sigma + it)|}{\log t}.$$

Below $H_{2005} = (32/205 + \varepsilon, 269/410 + \varepsilon)$ stands for Huxley's exponent pair from [6].

3. PRELIMINARY LEMMAS

Lemma 1. *Let $F: \mathbb{Z} \rightarrow \mathbb{C}$ be a multiplicative function such that $F(p^a) = f(a)$, where $f(n) \ll n^\beta$ for some $\beta > 0$. Then*

$$\limsup_{n \rightarrow \infty} \frac{\log F(n) \log n}{\log n} = \sup_{n \geq 1} \frac{\log f(n)}{n}.$$

Proof. See [14]. □

Lemma 2. *Let $f(t) \geq 0$. If*

$$\int_1^T f(t) dt \ll g(T),$$

where $g(T) = T^\alpha \log^\beta T$, $\alpha \geq 1$, then

$$I(T) := \int_1^T \frac{f(t)}{t} dt \ll \begin{cases} \log^{\beta+1} T & \text{if } \alpha = 1, \\ T^{\alpha-1} \log^\beta T & \text{if } \alpha > 1. \end{cases}$$

Proof. Let us divide the interval of integration into parts:

$$I(T) \leq \sum_{k=0}^{\log_2 T} \int_{T/2^{k+1}}^{T/2^k} \frac{f(t)}{t} dt < \sum_{k=0}^{\log_2 T} \frac{1}{T/2^{k+1}} \int_1^{T/2^k} f(t) dt \ll \sum_{k=0}^{\log_2 T} \frac{g(T/2^k)}{T/2^{k+1}}.$$

Now the lemma's statement follows from elementary estimates. □

Lemma 3. *For $\sigma \geq 1/2$ and for any exponent pair (k, l) such that $l - k \geq \sigma$ we have*

$$\mu(\sigma) \leq \frac{k + l - \sigma}{2} + \varepsilon.$$

Proof. See [7, (7.57)]. □

A well-known application of Lemma 3 is

$$(1) \quad \mu(1/2) \leq 32/205,$$

following from the choice $(k, l) = H_{2005}$. Another (maybe new) application is

$$(2) \quad \mu(3/5) \leq 1409/12170,$$

following from

$$(k, l) = \left(\frac{269}{2434}, \frac{1755}{2434} \right) = AB AH_{2005},$$

where A and B stands for usual A - and B -processes [8, Ch. 2].

Lemma 4. *Let $\eta > 0$ be arbitrarily small. Then for growing $|t| \geq 3$*

$$(3) \quad \zeta(s) \ll \begin{cases} |t|^{1/2-(1-2\mu(1/2))\sigma}, & \sigma \in [0, 1/2], \\ |t|^{2\mu(1/2)(1-\sigma)}, & \sigma \in [1/2, 1-\eta], \\ |t|^{2\mu(1/2)(1-\sigma)} \log^{2/3} |t|, & \sigma \in [1-\eta, 1], \\ \log^{2/3} |t|, & \sigma \geq 1. \end{cases}$$

More exact estimates for $\sigma \in [1/2, 1-\eta]$ are also available, e. g.

$$(4) \quad \mu(\sigma) \ll \begin{cases} 10(\mu(3/5) - \mu(1/2))\sigma + (6\mu(1/2) - 5\mu(3/5)), & \sigma \in [1/2, 3/5], \\ 5\mu(3/5)(1-\sigma)/2, & \sigma \in [3/5, 1-\eta], \end{cases}$$

Proof. Estimates follow from Phragmén–Lindelöf principle, exact and approximate functional equations for $\zeta(s)$ and convexity properties. See [15, Ch. 5] and [7, Ch. 7.5] for details. \square

Lemma 5. *For any integer r*

$$\max_{n \leq x} \lambda_r^{(e)}(n) \ll x^\varepsilon.$$

Proof. Surely $\lambda_r^{(e)}(n) \leq \lambda^{(e)}(n)$. By Lemma 1 we have

$$\limsup_{n \rightarrow \infty} \frac{\log \lambda^{(e)}(n) \log \log n}{\log n} = \sup_m \frac{\log \lambda(m)}{m} = \frac{\log 4}{5} =: c,$$

because $\lambda(m) \leq m-1$. It implies

$$\max_{n \leq x} \lambda^{(e)}(n) \ll x^{c/\log \log n} \ll x^\varepsilon.$$

\square

Lemma 6. *Let $L_r(s)$ be the Dirichlet series for $\lambda_r^{(e)}$:*

$$L_r(s) := \sum_{n=1}^{\infty} \lambda_r^{(e)}(n) n^{-s}.$$

Then for $r = 1, 2, 3, 4$ we have $L_r(s) = Z_r(s)G_r(s)$, where

$$(5) \quad Z_1(s) = \zeta(s)\zeta(3s)\zeta^2(5s),$$

$$(6) \quad Z_2(s) = \zeta(2s)\zeta^2(3s)\zeta(4s)\zeta^2(5s),$$

$$(7) \quad Z_3(s) = \zeta^2(3s)\zeta^2(4s)\zeta^4(5s),$$

$$(8) \quad Z_4(s) = \zeta^2(4s)\zeta^4(5s)\zeta^2(6s)\zeta^6(7s),$$

Dirichlet series $G_1(s)$, $G_2(s)$, $G_3(s)$ converge absolutely for $\sigma > 1/6$ and $G_4(s)$ converges absolutely for $\sigma > 1/8$.

Proof. Follows from the identities

$$\begin{aligned}
1 + \sum_{a \geq 1} \lambda^{(e)}(p^a) x^a &= 1 + x + x^2 + 2x^3 + 2x^4 + 4x^5 + 2x^6 + 6x^7 + O(x^8) \\
&= \frac{1 + O(x^8)}{(1-x)(1-x^3)(1-x^5)^2}, \\
1 + \sum_{a \geq 2} \lambda^{(e)}(p^a) x^a &= \frac{1 + O(x^6)}{(1-x^2)(1-x^3)^2(1-x^4)(1-x^5)^2}, \\
1 + \sum_{a \geq 3} \lambda^{(e)}(p^a) x^a &= \frac{1 + O(x^6)}{(1-x^3)^2(1-x^4)^2(1-x^5)^4}, \\
1 + \sum_{a \geq 4} \lambda^{(e)}(p^a) x^a &= \frac{1 + O(x^8)}{(1-x^4)^2(1-x^5)^4(1-x^6)^2(1-x^7)^6}.
\end{aligned}$$

□

Lemma 7. Let $\Delta(x)$ be the error term in the well-known asymptotic formula for $\sum_{n \leq x} \tau(a_1, a_2, a_3, a_4; n)$, let $A_4 = a_1 + a_2 + a_3 + a_4$ and let (k, l) be any exponent pair. Suppose that the following conditions are satisfied:

- (1) $(k + l + 2)a_4 < (k + l)a_1 + A_4$.
- (2) $2(k + l + 1)a_1 \leq (2k + 1)(a_2 + a_3)$.
- (3.1) $la_1 \leq ka_2$ and $(k + l + 1)a_1 \geq k(a_2 + a_3)$
- or
- (3.2) $la_1 \geq ka_2$ and $(l - k)(2k + 1)a_3 \leq (2l - 2k - 1)(k + l + 1)a_1 + (2k(k - l + 1) + 1)a_2$.

Proof. This is [9, Th. 3] with $p = 4$. □

Lemma 8.

$$m(\sigma) \geq \begin{cases} 4/(3 - 4\sigma), & 1/2 \leq \sigma \leq 5/8, \\ 10/(5 - 6\sigma), & 5/8 \leq \sigma \leq 35/54, \\ 19/(6 - 6\sigma), & 35/54 \leq \sigma \leq 41/60, \\ 2112/(859 - 948\sigma), & 41/60 \leq \sigma \leq 3/4, \\ 12408/(4537 - 4890\sigma), & 3/4 \leq \sigma \leq 5/6, \\ 4324/(1031 - 1044\sigma), & 5/6 \leq \sigma \leq 7/8, \\ 98/(31 - 32\sigma), & 7/8 \leq \sigma \leq 0.91591\dots, \\ (24\sigma - 9)/(4\sigma - 1)(1 - \sigma), & 0.91591\dots \leq \sigma \leq 1 - \varepsilon. \end{cases}$$

Proof. See [7, Th. 8.4]. □

4. MAIN RESULTS

Theorem 1.

$$\sum_{n \leq x} \lambda^{(e)}(n) = c_{11}x + c_{13}x^{1/3} + (c'_{15} \log x + c_{15})x^{1/5} + O(x^{1153/6073+\varepsilon}),$$

where c_{11} , c_{13} , c_{15} and c'_{15} are computable constants.

Proof. Lemma 6 and equation (5) implies that $\lambda^{(e)} = \tau(1, 3, 5, 5; \cdot) \star g_1$, where $\sum_{n \leq x} g_1(n) \ll x^{1/6+\varepsilon}$. Due to [8]

$$\begin{aligned}
\sum_{n \leq x} \tau(1, 3, 5, 5; n) &= x\zeta(3)\zeta^2(5) \operatorname{res}_{s=1} \zeta(s) + 3x^{1/3}\zeta(1/3)\zeta^2(5/3) \operatorname{res}_{s=1/3} \zeta(3s) + \\
&\quad + 5x^{1/5}\zeta(1/5)\zeta(3/5) \operatorname{res}_{s=1/5} \zeta^2(5s) + R(x).
\end{aligned}$$

To estimate $R(x)$ we use Lemma 7 with $a_1 = 1$, $a_2 = 3$, $a_3 = a_4 = 5$. Exponent pair $(k, l) = H_{2005}$ satisfies conditions 1, 2 and 3.2 and thus

$$R(x) \ll x^{(k+l+2)/(k+l+14)} = x^{1153/6073+\varepsilon}, \quad 1/6 < 1153/6073 < 1/5.$$

Now the convolution argument completes the proof. \square

Exponential totient function $\phi^{(e)}$ has similar to $\lambda^{(e)}$ Dirichlet series:

$$\sum_{n=1}^{\infty} \phi^{(e)}(n) = \zeta(s)\zeta(3s)\zeta^2(5s)H(s),$$

where $H(s)$ converges absolutely for $\sigma > 1/6$. Theorem 1 can be extended to this case without any changes, so

$$\sum_{n \leq x} \phi^{(e)}(n) = c_{11}x + c_{13}x^{1/3} + (c'_{15} \log x + c_{15})x^{1/5} + O(x^{1153/6073+\varepsilon}).$$

This improves the result of Pétermann [12], who obtained $\sum_{n \leq x} \phi^{(e)}(n) = c_{11}x + c_{13}x^{1/3} + O(x^{1/5} \log x)$.

Update from 23.05.2014: Recently Cao and Zhai [1, (1.13)] obtained $R(x) \ll x^{18/95+\varepsilon}$, which is better than both Pétermann's $x^{1/5} \log x$ and our $x^{1153/6073+\varepsilon}$.

Theorem 2.

$$\sum_{n \leq x} \lambda_2^{(e)}(n) = c_{22}x^{1/2} + (c'_{23} \log x + c_{23})x^{1/3} + c_{24}x^{1/4} + O(x^{1153/5586+\varepsilon}),$$

where c_{22} , c_{23} , c'_{23} and c_{24} are computable constants.

Proof. Similar to Theorem 1 with following changes: now by (6)

$$\lambda_2^{(e)} = \tau(2, 3, 3, 4; \cdot) \star g_2,$$

where $\sum_{n \leq x} g_2(n) \ll x^{1/6+\varepsilon}$. But

$$\begin{aligned} \sum_{n \leq x} \tau(2, 3, 3, 4; n) &= 2x^{1/2} \zeta^2(3/2) \zeta(2) \operatorname{res}_{s=1/2} \zeta(2s) + \\ &+ 3x^{1/3} \zeta(2/3) \zeta(4/3) \operatorname{res}_{s=1/3} \zeta^2(3s) + 4x^{1/4} \zeta(1/2) \zeta^2(3/4) \operatorname{res}_{s=1/4} \zeta(4s) + R(s). \end{aligned}$$

Again by Lemma 7 with $a_1 = 2$, $a_2 = a_3 = 3$, $a_4 = 4$, $(k, l) = H_{2005}$ we get

$$R(x) \ll x^{(k+l+2)/(k+l+12)} = x^{1153/5586+\varepsilon}, \quad 1/5 < 1153/5586 < 1/4.$$

\square

Theorem 3.

$$\begin{aligned} (9) \quad \sum_{n \leq x} \lambda_3^{(e)}(n) &= (c'_{33} \log x + c_{33})x^{1/3} + (c'_{34} \log x + c_{34})x^{1/4} + \\ &+ P_{35}(\log x)x^{1/5} + O(x^{1/6+\varepsilon}), \end{aligned}$$

where c_{33} , c'_{33} , c_{34} and c'_{34} are computable constants, P_{35} is a polynomial of degree 3 with computable coefficients.

Proof. Lemma 6 and equation (7) implies that $\lambda_3^{(e)} = z_3 \star g_3$, where z_3 is defined implicitly by

$$\sum_{n=1}^{\infty} z_3(n)n^{-s} = Z_3(s) = \zeta^2(3s)\zeta^2(4s)\zeta^4(5s),$$

and g_3 is a multiplicative function such that $\sum_{n \leq x} g_3(n) \ll x^{1/6+\varepsilon}$.

The main term at the right side of (9) equals to

$$M_3(x) := \left(\operatorname{res}_{s=1/3} + \operatorname{res}_{s=1/4} + \operatorname{res}_{s=1/5} \right) (\zeta^2(3s)\zeta^2(4s)\zeta^4(5s)x^s s^{-1}).$$

To obtain the desirable error term it is enough to prove that

$$\sum_{n \leq x} z_3(n) = M_3(x) + O(x^{1/6+\varepsilon}).$$

By Perron formula for $c := 1/3 + 1/\log x$ we have

$$\sum_{n \leq x} z_3(n) = \frac{1}{2\pi i} \int_{c-iT}^{c+iT} Z_3(s)x^s s^{-1} ds + O(x^{1+\varepsilon}T^{-1}).$$

Substituting $T = x$ and moving the contour of the integration till $[1/6 - ix, 1/6 + ix]$ we get

$$\sum_{n \leq x} f_3(n) = M_3(x) + O(I_0 + I_- + I_+ + x^\varepsilon),$$

where

$$I_0 := \int_{1/6-ix}^{1/6+ix} Z_3(s)x^s s^{-1} ds, \quad I_\pm := \int_{1/6 \pm ix}^{c \pm ix} Z_3(s)x^s s^{-1} ds.$$

Firstly,

$$I_+ \ll x^{-1} \int_{1/6}^c Z_3(\sigma + ix)x^\sigma d\sigma.$$

Let $\alpha(\sigma)$ be a function such that $Z_3(\sigma + ix) \ll x^{\alpha(\sigma)+\varepsilon}$. By (3) we have

$$\alpha(\sigma) \leq \begin{cases} (16 - 68\sigma)\mu(1/2) < 4/5, & \sigma \in [1/6, 1/5], \\ (8 - 28\sigma)\mu(1/2) < 3/4, & \sigma \in [1/5, 1/4], \\ (4 - 12\sigma)\mu(1/2) < 2/3, & \sigma \in [1/4, 1/3], \\ 0, & \sigma \in [1/3, c]. \end{cases}$$

This means that $I_+ \ll x^\varepsilon$. Plainly, the same estimate holds for I_- .

Secondly, it remains to prove that $I_0 \ll x^{1/6+\varepsilon}$. Here

$$I_0 \ll x^{1/6} \int_1^x Z_3(1/6 + it)t^{-1} dt$$

and taking into account Lemma 2 it is enough to show $\int_1^x Z_3(1/6 + it)dt \ll x^{1+\varepsilon}$. Applying Cauchy inequality twice we obtain

$$\begin{aligned} \int_1^x Z_3(1/6 + it)dt &\ll \left(\int_1^x |\zeta^4(1/2 + it)|dt \right)^{1/2} \times \\ &\times \left(\int_1^x |\zeta^8(2/3 + it)|dt \right)^{1/4} \left(\int_1^x |\zeta^{16}(5/6 + it)|dt \right)^{1/4} \ll \\ &\ll x^{(1+\varepsilon) \cdot 1/2} x^{(1+\varepsilon) \cdot 1/4} x^{(1+\varepsilon) \cdot 1/4} \ll x^{1+\varepsilon} \end{aligned}$$

since by Lemma 8 $m(1/2) \geq 4$, $m(2/3) \geq 8$ and $m(5/6) \geq 16$. \square

Theorem 4.

$$\begin{aligned} \sum_{n \leq x} \lambda_4^{(e)}(n) &= (c'_{44} \log x + c_{44})x^{1/4} + P_{45}(\log x)x^{1/5} + (c'_{46} \log x + c_{46})x^{1/6} + \\ &\quad + P_{47}(\log x)x^{1/7} + O(x^{C_4+\varepsilon}), \end{aligned}$$

where c_{44} , c'_{44} , c_{46} and c'_{46} are computable constants, P_{45} and P_{47} are computable polynomials, $\deg P_{45} = 3$, $\deg P_{47} = 5$,

$$(10) \quad C_4 = \frac{7863059 - \sqrt{13780693090921}}{85962240} = 0.134656\dots, \quad 1/8 < C_4 < 1/7.$$

Proof. We shall follow the outline of Theorem 3. Let us prove that for $c := 1/4 + 1/\log x$ we can estimate

$$I_+ := \int_{C_4+ix}^{c+ix} Z_4(s) x^s s^{-1} ds \ll x^{C_4+\varepsilon}$$

and

$$I_0 := \int_{C_4-ix}^{C_4+ix} Z_4(s) x^s s^{-1} ds \ll x^{C_4+\varepsilon}.$$

We start with $I_+ \ll x^{-1} \int_{C_4}^c Z_4(\sigma + ix) x^\sigma d\sigma$. Now let $\alpha(\sigma)$ be a function such that $Z_4(\sigma + ix) \ll x^{\alpha(\sigma)+\varepsilon}$. By (3) and (8) we have

$$\alpha(\sigma) \leq \begin{cases} (16 - 80\sigma)\mu(1/2) < 5/6, & \sigma \in [1/7, 1/6), \\ (12 - 56\sigma)\mu(1/2) < 4/5, & \sigma \in [1/6, 1/5), \\ (4 - 16\sigma)\mu(1/2) < 3/4, & \sigma \in [1/5, 1/4), \\ 0, & \sigma \in [1/4, c]. \end{cases}$$

So $\int_{1/7}^c Z_4(\sigma + ix) x^{\sigma-1} d\sigma \ll x^\varepsilon$ and the only case that requires further investigations is $\sigma \in [C_4, 1/7)$. Instead of (3) we apply (4) together with (1) and (2) to obtain

$$\alpha(\sigma) \leq \frac{1045018}{249485} - \frac{2459357}{99794}\sigma, \quad \sigma \in [1/8, 1/7],$$

which implies $\int_{C_4}^{1/7} x^{\alpha(\sigma)+\sigma-1} d\sigma \ll x^{C_4+\varepsilon}$ as soon as

$$C_4 \geq 1591066/12296785 = 0.129388\dots$$

Our choice of C_4 in (10) is certainly the case.

Let us move on I_0 and prove that $\int_1^x Z_4(C_4 + it) dt \ll x^{1+\varepsilon}$. For q_1, q_2, q_3, q_4 such that

$$(11) \quad 1/q_1 + 1/q_2 + 1/q_3 + 1/q_4 = 1 \quad \text{and} \quad q_1, q_2, q_3, q_4 \geq 1$$

by Hölder inequality we have

$$\begin{aligned} \int_1^x Z_4(C_4 + it) dt &\ll \left(\int_1^x |\zeta^{2q_1}(4s + it)| dt \right)^{1/q_1} \left(\int_1^x |\zeta^{4q_2}(5s + it)| dt \right)^{1/q_2} \times \\ &\quad \times \left(\int_1^x |\zeta^{2q_3}(6s + it)| dt \right)^{1/q_3} \left(\int_1^x |\zeta^{6q_4}(7s + it)| dt \right)^{1/q_4}. \end{aligned}$$

Choose

$$(12) \quad q_1 = m(4C_4)/2, \quad q_2 = m(5C_4)/4, \quad q_3 = m(6C_4)/2, \quad q_4 = m(7C_4)/6$$

One can make sure by substituting the value of C_4 from (10) into Lemma 8 that such choice of q_k satisfies (11). Thus we obtain

$$\int_1^x Z_4(C_4 + it) dt \ll x^{(1+\varepsilon)/q_1} x^{(1+\varepsilon)/q_2} x^{(1+\varepsilon)/q_3} x^{(1+\varepsilon)/q_4} \ll x^{1+\varepsilon},$$

which finishes the proof. \square

5. DECREASE OF C_4

In this section we obtain lower value of C_4 by improving lower bounds of $m(\sigma)$ from Lemma 8.

Estimates below depend on values of

$$(13) \quad \inf_{(k,l)} \frac{ak + bl + c}{dk + el + f},$$

where (k, l) runs over the set of exponent pairs and satisfies certain linear inequalities. A method to estimate (13) without linear constraints was given by Graham [5]. In the recent paper [10] we have presented an effective algorithm to deal with (13) under a nonempty set of linear constraints.

Let c be an arbitrary function such that $c(\sigma) \geq \mu(\sigma)$. Define θ by an implicit equation

$$2c(\theta(\sigma)) + 1 + \theta(\sigma) - 2(1 + c(\theta(\sigma)))\sigma = 0.$$

Finally, define

$$f(\sigma) = 2 \frac{1 + c(\theta(\sigma))}{c(\theta(\sigma))}.$$

Due to Lemma 3 one can take $c(\sigma) = \inf_{l-k \geq \sigma} (k + l - \sigma)/2$, where (k, l) runs over the set of exponent pairs. However even rougher choice of c leads to satisfiable values of f such as in [7, (8.71)].

Lemma 9. *Let $\sigma \geq 5/8$. Compute*

$$\begin{aligned} \alpha_1 &= \frac{4 - 4\sigma}{1 + 2\sigma}, & \beta_1 &= -\frac{12}{1 + 2\sigma}, & m_1 &= \frac{1 - \alpha_1}{\mu(\sigma)} - \beta_1, \\ \alpha_2(k, l) &= \frac{4(1 - \sigma)(k + l)}{(2 + 4l)\sigma - 1 + 2k - 2l}, & \beta_2(k, l) &= -\frac{4(1 + 2k + 2l)}{(2 + 4l)\sigma - 1 + 2k - 2l}, \\ m_2(k, l) &= \frac{1 - \alpha_2(k, l)}{\mu(\sigma)} - \beta_2(k, l), & m_2 &= \inf_{\alpha_2(k, l) \leq 1} m_2(k, l), \end{aligned}$$

where (k, l) runs over the set of exponent pairs. Then

$$m(\sigma) \geq \min(m_1, m_2, 2f(\sigma)).$$

Note that for $\sigma \geq 2/3$ the condition $\alpha_2(k, l) \leq 1$ is always satisfied.

Proof. Follows from [7, (8.97)] and from $T^\alpha V^\beta \ll TV^{\beta + (\alpha - 1)/\mu(\sigma)}$ for $\alpha < 1$ and $V \ll T^{\mu(\sigma)}$. \square

Substituting pointwise estimates of $m(\sigma)$ from Lemma 9 instead of segmentwise from Lemma 8 into (12) we obtain following result.

Theorem 5. *The statement of Theorem 4 remains valid for*

$$C_4 = 0.133437785 \dots$$

6. CONCLUSION

We have obtained nontrivial error terms in asymptotic estimates of $\sum_{n \leq x} \lambda_r^{(e)}(n)$ for $r = 1, 2, 3, 4$. Cases of $r = 1$ and $r = 2$ depend on the method of exponent pairs. Cases of $r = 3$ and $r = 4$ depend on lower bounds of $m(\sigma)$. Note that case of $r = 4$ may be improved under Riemann hypothesis up to $C_4 = 1/8$, because Riemann hypothesis implies $\mu(\sigma) = 0$ and $m(\sigma) = \infty$ for $\sigma \in [1/2, 1]$.

REFERENCES

- [1] Cao X., Zhai W. On the four-dimensional divisor problem of (a, b, c, c) type // *Funct. Approximatio, Comment. Math.* — 2013. — Vol. 49, no. 2. — P. 251–267.
- [2] Carmichael R. D. Note on a new number theory function // *Bull. Amer. Math. Soc.* — 1910. — feb. — Vol. 16, no. 5. — P. 232–238.
- [3] Erdős P., Pomerance C., Schmutz E. Carmichael's lambda function // *Acta Arith.* — 1991. — Vol. 58, no. 4. — P. 363–385.
- [4] Friedlander J. B., Pomerance C., Shparlinski I. E. Small values of the Carmichael function and cryptographic applications // *Cryptography and Computational Number Theory.* — Basel : Birkhauser Verlag, 2001. — Vol. 20 of *Progress in Computer Science and Applied Logic.* — P. 25–32.
- [5] Graham S. W. An algorithm for computing optimal exponent pair // *J. Lond. Math. Soc.* — 1986. — Vol. 33, no. 2. — P. 203–218.
- [6] Huxley M. N. Exponential sums and the Riemann zeta function V // *Proc. Lond. Math. Soc.* — 2005. — Vol. 90, no. 1. — P. 1–41.
- [7] Ivić A. The Riemann zeta-function: Theory and applications. — Mineola, New York : Dover Publications, 2003. — 562 p. — ISBN: 0486428133, 9780486428130.
- [8] Krätzel E. Lattice points. — Dordrecht : Kluwer, 1988. — 436 p. — ISBN: 9027727333, 9789027727336.
- [9] Krätzel E. Estimates in the general divisor problem // *Abh. Math. Semin. Univ. Hamb.* — 1992. — dec. — Vol. 62, no. 1. — P. 191–206.
- [10] Lelechenko A. V. Linear programming over exponent pairs // *Acta Univ. Sapientiae, Inform.* — 2013. — Vol. 5, no. 2. — P. 271–287.
- [11] Lelechenko A. V. Functions involving exponential divisors: bibliography. — 2014. — URL: <https://github.com/Bodigrim/expdiv-bibliography>.
- [12] Pétermann Y.-F. S. Arithmetical functions involving exponential divisors: Note on two papers by L. Tóth // *Ann. Univ. Sci. Budap. Rolando Eötvös, Sect. Comput.* — 2010. — Vol. 32. — P. 143–149.
- [13] Subbarao M. V. On some arithmetic convolutions // *The theory of arithmetical functions: Proceedings of the Conference at Western Michigan University, April 29 – May 1, 1971.* — Vol. 251 of *Lecture Notes in Mathematics.* — Berlin : Springer Verlag, 1972. — P. 247–271.
- [14] Suryanarayana D., Sita Rama Chandra Rao R. On the true maximum order of a class of arithmetic functions // *Math. J. Okayama Univ.* — 1975. — Vol. 17. — P. 95–101.
- [15] Titchmarsh E. C. The theory of the Riemann zeta-function / Ed. by D. R. Heath-Brown. — 2nd, rev. edition. — New-York : Oxford University Press, 1986. — 418 p. — ISBN: 0198533691, 9780198533696.

I. I. MECHNIKOV ODESSA NATIONAL UNIVERSITY

E-mail address: 1@dxdy.ru